Limit of small exits in open Hamiltonian systems

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The nature of open Hamiltonian systems is analyzed, when the size of the exits decreases and tends to zero. Fractal basins appear typically in open Hamiltonian systems, but we claim that in the limit of small exits, the invariant sets tend to fill up the whole phase space with the strong consequence that a new kind of basin appears, where the unpredictability grows indefinitely. This means that for finite, arbitrarily small accuracy, we can find *uncertain basins*, where any information about the future of the system is lost. This total indeterminism had only been reported in dissipative systems, in particular in the so-called intermingled riddled basins, as well as in the riddledlike basins. We show that this peculiar, behavior is a general feature of open Hamiltonian systems.

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I. INTRODUCTION

In the past few years a very particular property related to fractal basins has been shown in many dissipative systems. In some situations, and with a special symmetry, it is possible to find an attractor whose basin of attraction has the property that every point in the basin has pieces of another attractor's basin arbitrarily nearby. This type of basin is called *riddled basin*, because in some sense it is riddled with holes of another basin. When all basins in phase space are riddled by the rest, the basins are called *intermingled*. This remarkable result put in evidence in the context of dissipative systems the possibility to find a degree of uncertainty unknown at the moment, leading to extensive studies of this subject, both theoretical and experimental [1–4].

Furthermore, several reports have shown that in certain circumstances, fractal basins are mixed in such a way that from a practical point of view they resemble riddled basins, even if they do not verify all their mathematical properties. Different terminology has been used depending on the characteristics of these sets, some of them being *riddledlike* basins [5,6], *practical riddled* basins [7], *partially nearly riddled* basins [8], or *pseudoriddled* basins [9], to cite just a few. However, none of these approaches has analyzed yet the possibility of finding a similar phenomenon in Hamiltonian systems, and our work is focused in this direction.

When a test particle interacts with an open Hamiltonian system, it spends some time in a bounded area called the *scattering region* before crossing one of the existing exits and finally escaping to infinity (see Ref. [10] for a thorough study of this phenomenon, called *chaotic scattering*). The orbits that belong to the *nonattracting chaotic invariant set*, also known as the *chaotic saddle*, remain inside this region indefinitely, and the Lebesgue measure of this set is zero. As we are working with Hamiltonian systems, we cannot talk about basins of attraction, but we can define the exit basins as the sets of initial conditions that lead to a certain exit. Using this definition, we have studied in detail the evolution of exit basins in open Hamiltonian systems when the size of the exits decreases and tends to zero. And we have obtained the following striking result: *in the limit of small exits, the invariant sets of the system, that is, the chaotic saddle and its stable and unstable manifolds, tend to fill up physically the whole phase space*. A direct consequence of this result is that the basins suffer a total fractalization, becoming a new kind of fractal basins that we have named *uncertain basins* for its dramatic consequences on predictability. Furthermore, our result fully explains the tendency of the fractal dimension of the three invariant sets to the dimension of phase space, an idea first conjectured in Ref. [11]. Uncertain basins share with riddled and intermingled basins the main property of finding inside a ball of radius δ around *any point* of the basin, points that belong to other basins, being δ arbitrarily small. It is interesting to analyze the modified logistic map proposed in Ref. [6], as a precursor of our work. Thus, in this paper we claim that it is a general property shared by open Hamiltonian systems to possess an inherent uncertainty much stronger than expected, which in the limit of small exits makes a totally deterministic system become in practice a nondeterministic process, following the terminology used in Ref. [12].

II. DEPENDENCE OF THE EXIT BASINS ON THE SIZE OF THE EXITS

In order to present the real implications of this kind of basins, we will first show evidence of its existence in a paradigmatic hyperbolic system, and finally the same analysis will be developed for a nonhyperbolic system. In hyperbolic chaotic scattering, there are no Kolmogorov-Arnold-Moser (KAM) surfaces of quasiperiodic orbits, and all the periodic orbits are unstable. Nonetheless, in nonhyperbolic systems KAM surfaces are mixed with chaotic regions in the phase space. The existence or not of these surfaces brings important consequences to the dynamics of the system. In fact, in a hyperbolic environment, the survival probability of a test particle in the scattering region decays exponentially with time [i.e., $P(E,t) \sim e^{-t/\tau}$, where τ is the *average decay time* or *average transient lifetime*], while stickiness to KAM surfaces should make this decay algebraic [$P(E,t) \sim t^{-z}$] in nonhyperbolic systems (see Ref. [13], and references therein).

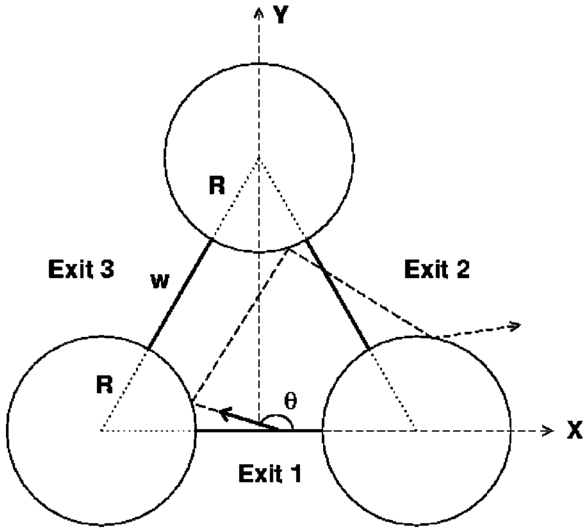

FIG. 1. Diagram of the three hard disk configuration. The only parameter that influences the system is w/R, where w represents the exit size and R the radius of the disks.

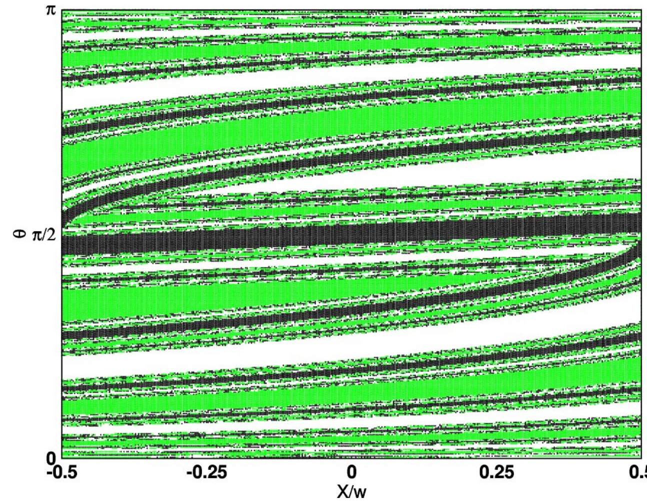

FIG. 2. Exit basin diagram for the three hard disk configuration, with 400×400 initial conditions (x,θ) and $w=0.2$. The initial conditions are plotted black if the orbit escapes through exit 1, gray for exit 2, and white for exit 3.

A. The hyperbolic system

The model we use for the hyperbolic case is a simple two-dimensional billiard consisting of three hard disks of radius R, whose centers are on the vertices of a triangle of side $L>2R$ (see Fig. 1). This configuration defines its scattering region as the bounded area between the disks and the triangle formed by their centers, and it has three exits of size $w=L-2R$. A typical test particle moves at constant speed in the scattering region, suffering elastic collisions with all the three disks, until it crosses one of the three exits of size w and escapes to infinity. This system was first studied in Ref. [14], extensively analyzed in the classical, semiclassical, and quantum regimes in Ref. [15] and examined in the context of microscopic deterministic diffusion in Refs. [16,17]. A nice review of the properties of its dynamics can be found in Ref. [18]. This model is one of the simplest and most general open Hamiltonian systems, and is a paradigm for low-dimensional chaotic scattering. For these reasons and for the sake of universality we use it here. Furthermore, in general terms, it is extremely complicated to verify rigorously the hyperbolicity of a system, and for this case it was done by Bunimovich and Sinai [19]. We have labeled the lower exit as exit 1, the right exit as exit 2, and the left exit as exit 3. Due to the triangular geometry of the system, it is important to remark that the only parameter that might influence its nature is w/R. For this reason, we have fixed the radius $R=1$ and have varied the exit size w as the control parameter to analyze the system. Note that other parameters such as the velocity of the test particle will not influence our results. We have situated the origin of coordinates in the middle point between the lower disks, that is, in the center of exit 1.

The exit basin diagram of an open Hamiltonian system gives us information about its dynamical behavior. In order to construct it for our system, we must follow a fine grid of initial conditions until they escape from the scattering region. The initial conditions that lead to exit 1 will belong to the exit 1 basin, while exits 2 and 3 are constructed in a similar way. Figure 2 shows the exit basin diagram for the system, when $w=0.2$. The color code we have chosen to plot this exit basin diagram is black for exit 1, gray for exit 2, and white for exit 3. The initial conditions are $y=0$, $x=(-w/2,w/2)$, and $\theta=(0,\pi)$, where θ is the shooting angle and is measured from the positive x axis in the counterclockwise sense. The initial conditions must be chosen carefully, because our results do not apply to test particles thrown from outside that do not enter the scattering region, since they do not even have chaotic behavior or an associated basin. Furthermore, as this diagram is two dimensional and phase space is three dimensional, our choice of initial conditions should include as many orbits as possible. In fact, it is easy to realize that our selection includes all the orbits that escape through exit 1 (in the opposite sense, but the system is time reversible) and due to the triangular symmetry of the system, we can say that these initial conditions represent all orbits that sooner or later escape from the system. Only the chaotic saddle and its unstable manifold are not included in this picture, and this is because they are formed by the orbits that remain for $t\rightarrow-\infty$ inside the scattering region and therefore are not represented by the orbits that enter this bounded region. However, this fact does not modify our results because their Lebesgue measure is zero. In the exit basin diagram we can see that the system is clearly fractal, as the basin boundaries are a nonsmooth mixture of all three colors. We have computed its fractality calculating the uncertainty dimension [20], and the result was $d=2.62\pm0.02$ for $w=0.2$ (where $d=2$ means nonfractality, and $d=3$ means total fractalization). Moreover, these basins possess the property of Wada [21], which means that any initial condition that is on the boundary of one exit basin is simultaneously on the boundary of all the other exit basins. However, it is fundamental to remark that there are large smooth black, gray, and white regions that belong to the interior of each basin, and contain no uncertainties over which exit is reached. We can say that those areas of phase space are *safe* [12].

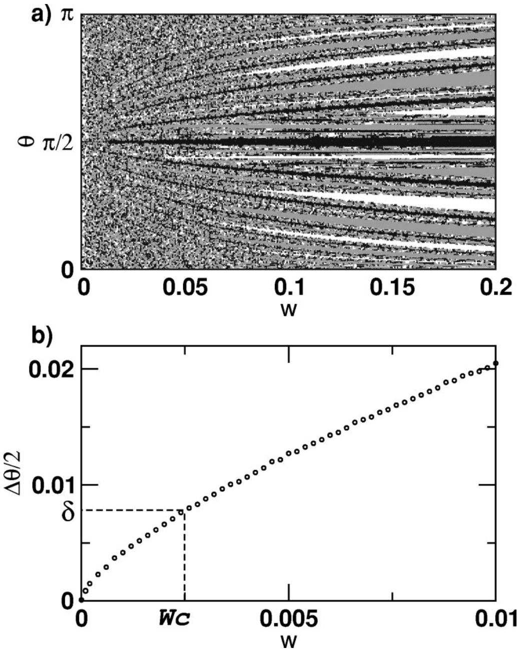

FIG. 3. (a) Evolution of the exit basin diagram for the three hard disk configuration, when the exit size w is varied. The initial conditions are $(x=0,y=0)$ and $\theta=(0,\pi)$. Exit 1 is plotted black, exit 2 is plotted gray, and exit 3 is plotted white. (b) Angular semiwidth of the largest open set of initial conditions for small exit sizes w.

Fractal basins are found both in Hamiltonian and dissipative systems, and they are composed of open sets separated by nonsmooth boundaries. Only when the distance between our initial condition and the basin boundary is shorter than the precision of our experiment, will we have trouble trying to predict its future behavior. In order to study the evolution of the unpredictability associated with the system when the exit size w is arbitrarily reduced, we have plotted in Fig. 3(a) the dependence of the exit basins on w. The test particle is always launched from $(x=0,\ y=0)$, and the range of shooting angles is $\theta=(0,\pi)$. In fact, this corresponds to a "1D slice" of initial conditions (the vertical line $x=0$ in Fig. 2), and it is plotted for a range of exit sizes $w=(0,0.2)$. We can clearly see that the fractal boundaries grow indefinitely, while the open sets of the three different basins shrink and tend to disappear in the limit of $w\rightarrow 0$. However, in order to give a more clear evidence of this fact, we have computed for low values of w the angular semiwidth $\Delta\theta/2$ of the black open set that belongs to basin 1 and is around $\theta=\pi/2$ in Fig. 3(a). We have chosen this safe region because it becomes for $w<0.1$ the biggest open set in phase space. This is done in Fig. 3(b), and it clearly confirms that the size of the biggest safe, connected open set of initial conditions in phase space tends to zero when w tends to zero. Obviously, the same result applies for the rest of open sets in phase space, which are smaller than this one.

B. The nonhyperbolic system

Most Hamiltonian systems are nonhyperbolic, and for this reason we have developed a similar analysis for a nonhyperbolic system, the Hénon-Heiles system, which has become, since it was first proposed in 1964 [22], a paradigm of simple Hamiltonian systems with very complicated dynamics. It has a triangular symmetry, and it is written as

$$H=\tfrac{1}{2}(\dot{x}^2+\dot{y}^2)+\tfrac{1}{2}(x^2+y^2)+x^2y-\tfrac{1}{3}y^3. \qquad (1)$$

This Hamiltonian has been extensively studied for the range of energy values above the escape energy $E_e=1/6=0.1666\ldots$, in Ref. [23]. When $E\leqslant 1/6$ all orbits are bounded, but for energies above this threshold value, the trajectories may escape from the scattering region and go on to infinity through three different exits. It was already evidenced in Ref. [23] that the dimension of all its invariant sets tends indeed to its maximum value (i.e., 3 in a 3D phase space) when $E\rightarrow E_e$ for $E>E_e$, and it was also shown that for $E<0.21$ a KAM torus exists. The quasiperiodic orbits that belong to a KAM torus never escape from the system, although they have energy to do so. In open Hamiltonian systems where the dynamics is defined by a certain potential, the size of the exits depends directly on the energy, and varying this value we can easily control w. These exits appear when the energy reaches the escape energy E_e. In Fig. 4(a) we have plotted the evolution of the exit basin diagram when the energy E is varied. The three different exit basins have been plotted in black, dark, gray and pale gray, while white has been used for the orbits belonging to the KAM torus. The initial conditions are $(x=0,\ \theta=0)$ and $y=(-0.5,1)$. The range of energy values is $E=(E_e=1/6,0.25)$.

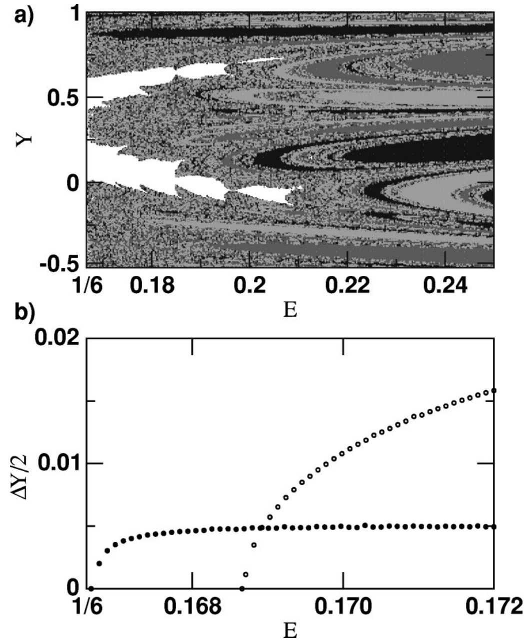

FIG. 4. (a) Evolution of the exit basin diagram for the Hénon-Heiles system, when the energy E is varied. The initial conditions are $(x=0,\theta=0)$, $y=(-0.5,1)$, and $E=(E_e=1/6=0.1666,0.25)$. Exit 1 is plotted black, exit 2 is plotted dark gray, and exit 3 is plotted pale gray. The KAM surface of quasiperiodic orbits is plotted white. (b) Vertical semiwidth of the two largest open sets of initial conditions for small values of the energy E. The dark dots represent the black open set around $y=0.87$, while the pale dots represent the black set around $y=0.98$ [almost unrecognizable in (a)].

Figure 4(b) shows the size of the two biggest open sets of initial conditions when the exit size tends to zero (in our case when $E \rightarrow E_e$ for $E > E_e$ backwards). The wide one is centered around $y = 0.87$ in Fig. 4(a), and clearly disappears before reaching the escape energy, following the behavior of most of the safe regions. The narrow one survives for values of the energy much closer to $E_e = 1/6$, although it is hardly recognizable around $y = 0.98$.

It is clear that we have obtained very similar results to those shown in Figs. 3(a) and 3(b) for the hyperbolic model. All open sets inside the basins indeed shrink and tend to disappear, and therefore these basins become completely fractalized and mixed in the limit of small exits. However, a remarkable difference between both kinds of systems has been detected. The KAM torus survives to the abrupt bifurcation that takes place when the energy crosses the value of the escape energy in the decreasing sense and the exits close. The reason is that these orbits, as they cannot leave the torus, do not even realize that the exits have disappeared, and in some sense it is possible to say that the torus is independent of the chaotic scattering phenomenon. In fact, the KAM torus takes part of the chaotic saddle, as its orbits remain inside the scattering region for both $t \rightarrow \infty$ and $t \rightarrow -\infty$. Consequently, the KAM torus remains "alive" when the exits become arbitrarily small, and therefore the fractalized basins only fill up the phase space that is not occupied by the KAM surfaces.

III. EXISTENCE AND NATURE OF UNCERTAIN BASINS

The computational evidence obtained for hyperbolic and nonhyperbolic systems leads us to the main result of our work: *For all points P in the escaping phase space of an open Hamiltonian system, and for all $\delta > 0$ (precision of the experiment), there exists a critical size of the exits $w_c > 0$ such that for all $w \leqslant w_c$ we can find a point P' in a ball centered in P and radius δ that belongs to a different basin than P* [see Fig. 2(b)]. We name *escaping phase space* the whole phase space for hyperbolic systems, and the phase space not occupied by KAM tori for nonhyperbolic systems. We propose that this result is applicable for all kinds of open Hamiltonian systems, even those in which the size of the exits w is not an available parameter (such as the ones defined by potentials, for example). The reason is that there is always a direct relation between the size of the exits w and the main parameter in those systems, the energy E.

All numerical or real experiments have an unavoidable finite precision associated to the choice of initial conditions. New techniques or more developed instruments might increase this precision, but will never make an initial condition infinitely accurate. This fact was emphasized in Refs. [5,6], where riddledlike basins were presented in the context of transient chaos and permanent chaos, respectively. For this reason, given a finite δ, arbitrarily small, we are sure that if the size of the exits is sufficiently small, all the open sets with points of a certain basin (i.e., the safe regions) will be smaller than this threshold value. Then, we will not be able to ascertain which basin any initial condition belongs to, and therefore we will not have any information about its future

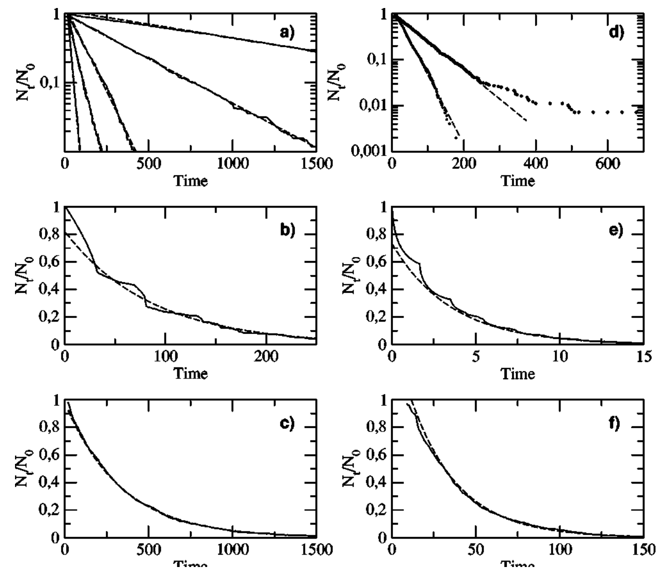

FIG. 5. Fraction of remaining orbits N_t/N_0 in a function of the time t for the three-disk configuration (a)–(c) and the Hénon-Heiles system (d)–(f). For the three-disk configuration: (a) $w = 1, 0.05, 0.02, 0.005, 0.001$ (from left to right). (b) $w = 0.02$; (c) $w = 0.005$. For the Hénon-Heiles system: (d) $E = 0.215$ (without KAM torus, in the hyperbolic regime), $E = 0.19$ (with KAM torus, in the nonhyperbolic regime). (e) $E = 0.4$; (f) $E = 0.215$.

behavior. The striking fact is that for hyperbolic systems this total uncertainty *is applied to all points in phase space*. For nonhyperbolic systems, however, if we choose an initial condition inside a KAM torus, we know that its future behavior will be a bounded orbit inside the scattering region, and the same stands for the initial conditions nearby. Then, we can still affirm that arbitrarily close to any initial condition that belongs to one basin, there are initial conditions that belong to other basins, but we cannot talk about absolute uncertainty in nonhyperbolic systems, as the KAM tori remain then as *deterministic islands* surrounded by a *random sea*.

As it has just been commented, the existence of uncertain basins incapacitates us from knowing in advance the exit chosen by the particle to escape, that is, the exit basin diagram becomes a useless tool. However, this does not mean that there are no other methods for studying dynamical systems that can still give us some information about the nature of the orbits. As a consequence of the previous discussion, it is in the context of predictability of the final state of the system that we propose that uncertain basins imply total indeterminism, and that in essence the system becomes random. For example, the different decay of the survival probability in hyperbolic and nonhyperbolic systems should be maintained in the limit of small exits. In order to show this fact, we have studied the evolution of the survival probability of both models when the size of the exits tends to zero. In Fig. 5, we have plotted the evolution of the fraction of remaining orbits in the scattering region N_t/N_0 with time, where N_t is the number of remaining orbits after a time t and N_0 is the number of initial orbits. The initial conditions are the barycenter of the triangle and $\theta = (0, \pi)$ for the three-disk configuration, and $x = 0$, $\theta = 0$, and $y = (-0.5, 1)$ for the

Hénon-Heiles system [see Fig. 4(a)]. Figure 5(a) shows N_t/N_0 for $w=1$, 0.05, 0.02, 0.005, and 0.001 for the three-disk configuration, and Fig. 5(b) shows the same quantity for $E=0.215$ and 0.19 for the Hénon-Heiles system. In both figures N_t/N_0 is plotted in a logarithmic scale, so the exponential approximations (dashed lines) are straight lines. In Fig. 5(a) we can clearly see that the exponential approximation that should be expected for a hyperbolic system is very accurate, no matter how small the exits are. In Fig. 5(d) we have plotted the same curve for the Hénon-Heiles system, and two different values of the energy. For $E=0.215$ there are no KAM tori [see Fig. 4(a)], and therefore we are in the hyperbolic regime of the Hénon-Heiles Hamiltonian. For this reason, the exponential approximation is also very accurate. However, for $E=0.19$, there is a KAM torus that fills the 14.5% of the initial conditions used to paint these curves, and the exponential approximation fits until only 3% of the escaping orbits remain in the system. Note that to paint this curve, the quasiperiodic orbits of the KAM torus were not included in N_0. Furthermore, it is important to say that the exponential approximation of $E=0.19$ was done taking into account only $t<250$, as it is obvious that the rest does not follow it at all. The results obtained for $E=0.19$ mean that most of the orbits are not sensible to the existence of the KAM torus, and only the orbits that start very close to it suffer the expected *stickiness* that makes them escape with a slower rate than the exponential. It is remarkable that this new rate does not fit very accurately the expected algebraic decay, and therefore we suppose that what we have in this case is a complicated mixture of both phenomena.

For high values of the size of the exits, that is, for high w and E [see Figs. 5(b) and 5(e)], the existence of big open sets of initial conditions makes the curves start with an irregular pattern formed by several components, while the exponential approximation is only suitable for high times (for a thorough explanation of this phenomenon, see Ref. [23]). However, when the exits get smaller and smaller, the open sets shrink and in consequence the survival probabilities tend to fit the exponential approximation very accurately even for small times. This is shown in Figs. 5(c) and 5(f), where $w=0.005$ and $E=0.215$, respectively. According to these curves, we might affirm that the accuracy of the exponential approximation for small times is a measure of the fractalization of phase space, and it gives us an idea of how deterministic systems lose in the limit some of their particular characteristics, to obtain a probabilisticlike nature.

An important goal of this paper is to explain, from a qualitative point of view, the consequences of the existence of uncertain basins in a Hamiltonian system. In this context, the total fractalization of phase space presented in this work can be explained as follows. When a set of initial conditions hits a hard disk (or the wall of a potential), it is divided in several sets of orbits, some escape, and some remain inside the scattering region. After the next hit the *survival* sets are again divided in smaller groups, some of them escaping and some of them hitting another disk. This situation is repeated *ad eternum*, and is responsible for the existence of a Cantor set of orbits that never escape from the system. In fact, the orbits that separate the sets that escape through two different exits will remain in the system forever as boundary points, constituting the stable manifold of the chaotic saddle. The *escape time function* of a set of initial conditions is defined as the time that each initial condition takes to escape from the system. In the interior of every open set belonging to the exit basin diagram of a generic open Hamiltonian system, the escape time function is continuous and has no singularities, while it tends to infinity in its boundary. In some sense, the orbits that belong to these sets escape *all together* and through the same exit, as we are sure that the neighborhood of each orbit leads to the same exit. The size of these sets obviously depends on the size of the exits. If the exits decrease, these open sets will also shrink, since less and less orbits will be able to pass through the exits *as a group*. For this reason, in the limit of small exits the open sets in each basin tend to have zero volume. Intuitively, we could say that in the limit only one orbit can escape at a time, the *safe regions* have become points, and therefore all basins, totally fractalized, tend to coincide with their own boundary. Then, the number of orbits that remain in the system forever will increase indefinitely, making the stable manifold of the invariant set tend to fill up the whole phase space. The unstable manifold will behave in the same way, because both manifolds are symmetric, and also their intersection, which is the nonattracting chaotic set.

Moreover, the tendency of the fractal dimension of the invariant sets to the dimension of phase space when the exits tend to zero is clearly explained as a corollary of our results. The *uncertainty* dimension [20] is calculated as a function of the variation of the number of *uncertain orbits* when the grid of initial conditions is changed. The uncertain orbits are defined as the orbits that tend to one exit while their closest neighbors tend to different exits. If the size of all open sets in the basin diagram goes to zero, all points in the escaping phase space become uncertain at all scales, making the dimension of the stable manifold of the chaotic saddle tend to its maximum value, which coincides with that of the phase space. This behavior must also be shown by the other two invariant sets, due to the relationship that exists among them. The uncertainty exponent α is defined as $\alpha=D-d$, D being the phase space dimension and d the uncertainty dimension. For this reason, the uncertainty exponent α of an uncertain basin should tend to zero in the limit of small exits.

Finally, in order to give visual evidence of the uncertain basins discussed in this paper, we have plotted them for both examples, the hyperbolic and the nonhyperbolic system. Figure 6(a) shows the exit basin diagram for the three hard disk configuration, for a very low value of the exit size w, in particular, $w=0.001$. As expected, the basins are far worse defined than in Fig. 2, where $w=0.2$, and no open sets are now recognizable. The picture is a mixture of dots that belong to all three basins. In order to show that all three basins tend to fill up the whole phase space at all scales, we have plotted in Figs. 6(b), 6(c), and 6(d) basin 1, basin 2, and basin 3, respectively. They show clearly the unstoppable growth of the fractal region, as well as the tendency of each basin to become its own boundary. Each pixel in Fig. 6 has a vertical size of $\Delta\theta=\pi/200=0.016$, and according to Fig. 3(b), the biggest open set in phase space will have a vertical

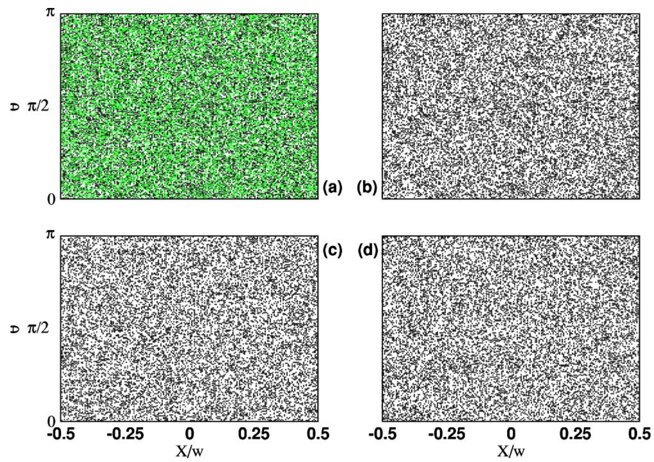

FIG. 6. (a) Exit basin diagram for the three hard disk configuration, with 200×200 initial conditions (x,θ) and $w=0.001$. Exit 1 is plotted black, exit 2 is plotted gray, and exit 3 is plotted white. (b), (c), and (d) depict exit basins 1, 2, and 3, respectively. All three basins tend to coincide with their own boundaries and fill up the phase space when the exit size w decreases.

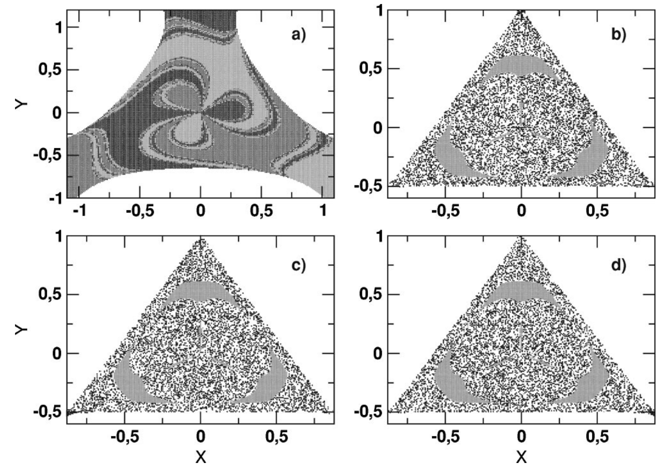

FIG. 7. (a) Exit basin diagram for the Hénon-Heiles system, with 200×200 initial conditions (x,y) and $E=0.3$. Exit 1 is plotted black, exit 2 is plotted dark gray, exit 3 is plotted pale gray, and the orbits not allowed for this value of the energy are plotted white. (b), (c), and (d) Exit basins 1, 2, and 3, respectively, for $E=0.1675$. The KAM torus that exists for $E<0.21$ is plotted gray. All three basins tend to coincide with their own boundaries and fill up the phase space (except the KAM torus) when the energy E decreases and tends to $E_e=0.1666$.

diameter of around $\Delta\theta=0.008$. Therefore, we can assure that all open sets are smaller than one pixel. If the resolution of this picture coincided with our experimental precision, this exit basin diagram would be useless to predict the future of the system. We would only know that every initial condition has a probability 1/3 to reach each exit.

The fractal dimension calculated as the uncertainty dimension for this value of $w=0.001$ is $d=2.998\pm0.005$, and therefore the uncertainty exponent $\alpha=D-d=0.002\pm0.005$, where D is the dimension of phase space (3 in our case). In Refs. [5,6] it is emphasized that the uncertainty exponent $\alpha\approx0$ for riddled basins, and several reported values are $\alpha=0.017$ [2], $\alpha=0.003$ [24], and $\alpha=0.0089$ [25].

In the same way, Figs. 7(b), 7(c), and 7(d) show basins 1, 2, and 3, respectively, for the Hénon-Heiles system, for a value of the energy $E=0.1675$ (very close to the escape energy $E=0.1666$). We have plotted in Fig. 7(a) the same exit basins but with a higher energy, $E=0.3$, and if we compare this picture with the other three, we can observe that the safe regions that can be easily recognized in (a) have clearly disappeared in (b)–(d). As it was shown in Fig. 6, there are no recognizable open sets or defined structures in these uncertain basins, apart from the KAM torus. It must also be observed that the KAM torus in (b)–(d) does not appear in (a).

IV. DISCUSSION

In order to compare the phenomenon of uncertain basins in Hamiltonian systems with the already known of riddled basins for dissipative systems, it might be useful to observe again Figs. 6 and 7. A riddled basin is a basin where all points have pieces of another basin arbitrarily nearby. It coincides with its own boundary, and a consequence of this definition is that riddled basins do not have any open sets inside. From a practical point of view, the similarities between uncertain and riddled basins are striking. As it was already commented, we can always find a threshold value for the size of the exits under which we are sure that the open sets (or the interior) of the basins are smaller than the unavoidable precision of the experiment. For that reason, for a finite, arbitrarily small accuracy, there is no physical way to distinguish the shrinking open sets of uncertain basins from the totally disconnected points in riddled basins. In fact, we have shown that the uncertainty dimension of uncertain basins tends to that of phase space (or the uncertainty exponent tends to zero), which is the value that would be measured in a typical riddled basin. On the other hand, there is a basic mathematical difference between both concepts. It is the fact that, formally speaking, the uncertain basins do not coincide exactly with their own boundary, as their open sets are indeed arbitrarily small but of positive size. In the limit of $w=0$ (no exits) the size of these sets is strictly 0, but then basins 1, 2, and 3 disappear and the Lebesgue measure of the chaotic saddle suffers a discontinuous jump from 0 to a positive value, the measure of phase space. In some sense, the transition between an open and a closed Hamiltonian system can be understood as a bifurcation in which the chaotic saddle suddenly fills up the whole phase space, making impossible the escape of any orbit.

In conclusion, we have presented a thorough analysis of the bifurcation that takes place when the size of the exits of open Hamiltonian systems tends to zero. We have seen that the exit basins show a peculiar behavior, very similar to that of riddled basins in dissipative systems. They suffer a total fractalization, tending to become their own boundaries while the dimension of the invariant sets coincides with that of phase space. Furthermore, these invariant sets tend to fill up the whole phase space for both hyperbolic and nonhyperbolic systems. This behavior makes any prediction based on the dynamical analysis of the exit basin diagrams absolutely useless. Totally deterministic systems become, in practice, random processes, and up to now, such degree of uncertainty was unknown in Hamiltonian systems. Finally, we believe that an experimental optical verification of our results, based on a triangular configuration of curved mirrors in which the distance between them could be modified, might be a simple and promising task.

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